Anisotropic conductivity of disordered 2DEGs due to spin-orbit interactions

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(Dated: March 31, 2008)

We show that the disorder-averaged conductivity tensor of a disordered two-dimensional electron gas becomes anisotropic in the presence of both Rashba and Dresselhaus spin-orbit interactions (SOI). This anisotropy is a mesoscopic effect and vanishes with vanishing charge dephasing time. Using a diagrammatic approach including zero, one, and two-loop diagrams, we show that a consistent calculation needs to go beyond a Boltzmann equation approach. In the absence of charge dephasing and for zero frequency, a finite anisotropy $\sigma_{xy} \propto e^2/p_\mathrm{F}lh$ arises even for infinitesimal SOI, where p_F is the Fermi momentum and l is the mean free path of an electron.

I. INTRODUCTION

The interplay between spin and charge coherence effects in mesoscopic semiconductors produces interesting transport phenomena[1,2,3]. They are based on the spin-orbit interaction (SOI), such as Rashba[4] or Dresselhaus[5] type, which establishes a coupling between orbital and spin degrees of freedom of the electron.

One major effect of SOI on the conductivity of a disordered semiconductor is the sign reversal of weak localization effects. In this respect the influence of the SOI is similar to that of a magnetic field: it increases the conductivity. For Rashba and Dresselhaus SOI and in the presence of a magnetic field, this "weak antilocalization" effect has been studied in a number of papers[6,7,8,9].

In the absence of magnetic fields, the Rashba SOI cannot violate isotropy of the energy spectrum. In this case, the conductivity tensor $\sigma_{\alpha\beta}$ is invariant under rotation of the coordinate system (CS) by $\pi/2$ and simultaneous sign change of the SOI. However, this sign is irrelevant for the conductivity (see Appendices), and thus $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$. In addition, due to time reversal invariance, $\sigma_{xy} = \sigma_{yx}$, so that $\sigma_{\alpha\beta}$ remains isotropic, i.e. $\sigma_{xy} = 0$. This reasoning is no longer valid if rotational invariance of the spectrum is broken, e.g., by a Zeeman term which violates both time-reversal and rotational symmetries, leading to a finite anisotropy of the conductivity[10].

A similar situation arises even without magnetic fields but when different types of SOI (such as Rashba and Dresselhaus) are present: the spectrum becomes anisotropic[11] so that one can expect anisotropy of the conductivity. However, differently from before[10], a system with only SOI remains invariant under time reversal. Nevertheless, we demonstrate below for a disordered two-dimensional electron gas (2DEG) that breaking of the rotation symmetry of the spectrum alone leads to an anisotropic conductivity.

Previous calculations in such a system were based on the Boltzmann equation[11,12], with the outcome[13] that the conductivity does get enhanced by the SOI, but remains isotropic even when both Rashba and Dresselhaus terms are present.

However, in the presence of phase coherence, both for charge and spin, the standard Boltzmann approach is no longer sufficient. Indeed, this approach correctly describes contributions to $\sigma_{\alpha\beta}$ of the order of the Drude conductivity $\sigma_\mathrm{D} = e^2 p_\mathrm{F} l/2h$, but, as is well-known, it already 'lacks accuracy' to describe weak localization corrections $\sim \sigma_\mathrm{D}\hbar/p_\mathrm{F}l \ll \sigma_\mathrm{D}$ (here p_F and l are Fermi momentum and mean free path of the electrons, *resp.*). We will see that the required accuracy for finding the anisotropy of the conductivity is even higher.

In the diagrammatic approach[14] used below, the leading contribution of a diagram to $\sigma_{\alpha\beta}$ is of the order of $\sigma_\mathrm{D}/(p_\mathrm{F}l/\hbar)^n$, where n is the number of loops built by cooperon and diffuson lines in the diagram. (This is often called "loop expansion".) Thus, the zeroth order is represented by two diagrams: the Drude "bubble" and the "vertex correction", which are often referred to as "zero loop approximation" (ZLA)[15], indicating that these two diagrams have no loops made of cooperon and/or diffuson lines. One can easily check that, in leading order, these ZLA-diagrams are SOI-independent (the "bubble" gives σ_D, while the "vertex correction" vanishes).

The SOI-dependent contribution to $\sigma_{\alpha\beta}$, coming from the ZLA, is *isotropic* and on the order $\sigma_\mathrm{D}/(p_\mathrm{F}l/\hbar)^2$. We will see that this is of the same order as contributions from diagrams with two loops. Thus, a consistent calculation (i.e. a systematic expansion in powers of $\hbar/p_\mathrm{F}l$) requires consideration of all the diagrams having zero, one, and two loops. Below we demonstrate that the isotropic conductivity, obtained from the Boltzmann equation[12], corresponds to the ZLA. The inconsistency of the ZLA, and thus of the Boltzmann equation, has been pointed out before[15] in the context of the spin-Hall effect.

Taking all relevant diagrams systematically into account, we find that the *conductivity has finite anisotropic components* given by Eq. (16), or, when expressed in terms of charge and spin dephasing times, by Eq. (18). In the fully phase coherent limit, a finite anisotropy $\sigma_{xy} \propto e^2/2\pi p_\mathrm{F}l$ exists even for infinitesimal SOI.

The paper is organized as follows. In Sec. II, we calculate matrix elements of the diffuson at zero frequency; the result leads to the cancellation of the anomalous part of the velocity operator. Then, in Sec. III we realize that the (most commonly used) zero loop approximation

(ZLA) gives an isotropic contribution to the conductivity tensor, similar to the result of the Boltzmann equation[12]. Then, we demonstrate that this ZLA result is incomplete in a given order $\hbar/p_F l$, and higher order diagrams have to be considered. In Sec. IV we obtain the general form of the expansion for the anisotropic part of the conductivity. The main results are obtained in Sec. V, where we calculate the leading contribution to the anisotropy determined by two-loop diagrams and give estimates for real samples.

II. KUBO FORMULA AND VERTEX RENORMALIZATION

Since the considered system is invariant under time reversal, $\sigma_{\alpha\beta}$ is symmetric and can thus be diagonalized. We will proceed with calculations in the CS rotated by $\pi/4$ with respect to the original one, where the 2D conductivity tensor is diagonal.

In linear-response theory, the conductivity tensor is given by the Kubo-Greenwood formula:

$$\sigma_{\alpha\beta} = \frac{e^2}{h} \overline{\text{Tr}\left[\hat{v}_\alpha \hat{G}_R \hat{v}_\beta \hat{G}_A\right]}, \quad \alpha,\beta = x,y, \qquad (1)$$

where the overbar indicates averaging over the different disorder realizations, $\hat{v}_\alpha = \frac{i}{\hbar}[\hat{H}, r_\alpha]$ is a component of the velocity operator, and $\hat{G}_{R/A} = [E_F - \hat{H} \pm i0]^{-1}$ with E_F being the Fermi energy (the derivation of (1) is analogous to the one in[15]). The Hamiltonian in our (rotated) CS reads

$$\hat{H} = \frac{\hat{p}^2}{2m} + V_s + U(\mathbf{r}),$$
$$V_s = (a-b)\tilde{\sigma}_1 \hat{p}_y - (a+b)\tilde{\sigma}_2 \hat{p}_x, \qquad (2)$$

where a and b are the amplitudes of Rashba and Dresselhaus SOI, $\tilde{\sigma}_{1,2}$ are expressed in terms of Pauli matrices as $\tilde{\sigma}_{1,2} = (\sigma_2 \pm \sigma_1)/\sqrt{2}$, and $U(\mathbf{r})$ is a short-range impurity potential, $\overline{U(\mathbf{r})U(\mathbf{r}')} = \hbar^3(m\tau)^{-1}\delta(\mathbf{r}-\mathbf{r}')$ (we use notations similar to[15]). The conductivity (1) is not affected by a simultaneous sign-reversal of a and b; thus, without loss of generality, we can assume that $a+b > 0$.

In the absence of SOI, the disorder-averaged retarded Green function (GF) is given by[14]

$$G_R^{(0)} = \left[G_A^{(0)}\right]^\dagger = \sigma_0 \left[E_F - \frac{p^2}{2m} + \frac{i\hbar}{2\tau}\right]^{-1}, \qquad (3)$$

where σ_0 is the 2×2 unity matrix (due to spin degrees of freedom), and τ is the mean time between collisions of an electron off impurities. The presence of the SOI modifies (3) as follows:

$$G_R = G_R^{(0)} \sum_{n\geq 0} \left(V_s G_R^{(0)}\right)^n = \left[\left(G_R^{(0)}\right)^{-1} - V_s\right]^{-1}. \qquad (4)$$

The disorder-averaging of (1) produces an infinite number of diagrams classified according to the number of loops composed by cooperon and diffuson lines. Each of these diagrams contains two velocity vertices, both[16] being renormalized by the vertex correction[15], so that their "anomalous" (i.e., SOI-dependent) part cancels:

$$\hat{\mathbf{v}} = \frac{\hat{\mathbf{p}}}{m} + \begin{pmatrix} -(a+b)\tilde{\sigma}_2 \\ (a-b)\tilde{\sigma}_1 \end{pmatrix},$$
$$\hat{\tilde{v}}_\alpha = \hat{v}_\alpha + \sum_{\gamma=1}^3 \tilde{\sigma}_\gamma D^{\gamma\gamma} \text{Tr}_{\mathbf{p}}\left[\tilde{\sigma}_\gamma G_R(\mathbf{p})\hat{v}_\alpha G_A(\mathbf{p})\right] = \frac{\hat{p}_\alpha}{m}, \qquad (5)$$

where

$$(D^{11}, D^{22}, D^{33}) =$$
$$= \frac{\hbar}{m\tau}\left(1 + \frac{1+K}{(x_a+x_b)^2}, 1 + \frac{1+K}{(x_a-x_b)^2}, \frac{K}{K-1}\right) \qquad (6)$$

are the components of the diffuson at zero momentum, $K = \sqrt{[1+(x_a+x_b)^2][1+(x_a-x_b)^2]}$, and we have introduced dimensionless Rashba and Dresselhaus amplitudes, $x_a = 2p_F a\tau/\hbar$ and $x_b = 2p_F b\tau/\hbar$. [Note that (5) is not exact at finite frequency.]

III. THE ZERO-LOOP APPROXIMATION

In the ZLA, the conductivity is given by two diagrams – the "bubble" and the "vertex correction"[15], which, when being summed, result in the following SOI-dependent correction to the conductivity[17]:

$$\delta\sigma_{\alpha\beta} = \frac{e^2}{h} \text{Tr}_{\mathbf{p}}\left[\hat{\tilde{v}}_\alpha G_R(\mathbf{p})\hat{v}_\beta G_A(\mathbf{p})\right.$$
$$\left. - \hat{\tilde{v}}_\alpha G_R^{(0)}(\mathbf{p})\hat{v}_\beta G_A^{(0)}(\mathbf{p})\right] = \frac{e^2}{h}\frac{x_a^2+x_b^2}{2p_F l/\hbar}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (7)$$

where $G_{R/A}^{(0)}$ is given in (3) and $\text{Tr}_{\mathbf{p}} \equiv \int \frac{d^2p}{(2\pi)^2}$. The result (7) has been confirmed (up to a missing factor 4) in[12] by using the Boltzmann equation approach; so one can conclude that ZLA corresponds to the result of the Boltzmann equation with the simplest (and most common) form of the collision integral. However, since (7) is of the order of $\sigma_D/(p_F l/\hbar)^2 \ll \sigma_D$, the calculation in the ZLA (as well as the Boltzmann equation[12]) is incomplete.

To re-enforce this point, let us consider the special case when $a = \pm b$. Then, all four operators under the Tr in (1) can be diagonalized by a momentum-independent unitary transformation, and one can see that the SOI-dependent correction to the conductivity vanishes already before the disorder averaging (see Appendix B). This is just further evidence that the expression (7) is incomplete; other contributions must be taken into account to cancel it when $a = \pm b$.

IV. CONTRIBUTION OF HIGHER ORDERS

According to the loop expansion[15], diagrams having one (weak localization[6,7]) and two loops may produce
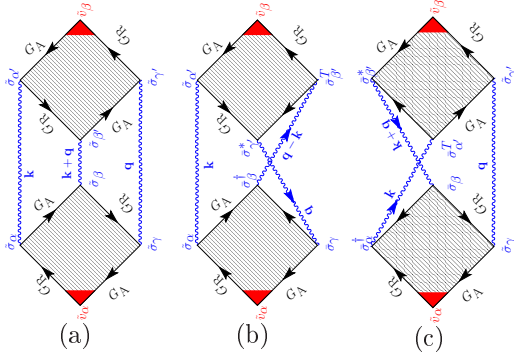
FIG. 1: Three relevant two-loop diagrams (out of nine) for the conductivity tensor $\sigma_{\alpha\beta}$. The matrix vertices $\tilde{\sigma}$ and $\bar{\sigma}$ are written assuming clockwise (in lower Hikami boxes) and counterclockwise (in upper Hikami boxes) direction of writing traces (Tr). Each diagram contains small-momentum singularities, which cancel each other in accordance with the theorem from[18].

contributions to $\sigma_{\alpha\beta}$ of the same or even larger order than (7). There exist one diagram with one loop and nine diagrams with two loops.

From now on we assume that the SOI and the spectrum anisotropy due to it are small:

$$x = \sqrt{x_a^2 + x_b^2} \ll 1, \ \delta = \frac{2ab}{a^2 + b^2}, \ |\delta| \ll 1. \quad (8)$$

We have chosen x and δ as expansion parameters because they provide a uniform expansion for the conductivity (9).

The anisotropic part of the conductivity tensor is given by an expansion in inverse powers of $p_F l/\hbar \gg 1$:

$$\sigma_{xx} - \sigma_{yy} = 2\frac{e^2}{h} \sum_{r \geq 0} \frac{1}{(p_F l/\hbar)^{r-1}} \sum_{m,n \geq 0} S^r_{mn} x^m \delta^{2n+1}, \quad (9)$$

where we used (C6).

By calculating the conductivity in the ZLA we have checked that $S^0_{mn} = 0 \ \forall m, n$. From the properties of the weak localization (WL) diagram (which is the only one that could contribute to S^1_{00}), one can easily show that $S^1_{00} = 0$ as well. Thus, in the limit when $p_F l\delta^2/\hbar < 1$, the leading anisotropic contribution is given by the term $\propto S^2_{00}$ in the expansion (9), which we calculate below.

V. DIAGRAMS WITH TWO LOOPS

We have checked that, like the ZLA, the WL diagram does not contribute to S^2_{00}. The same is true for 6 (out of 9) two-loop diagrams. The remaining 3 diagrams, which we calculate below, are depicted in Fig. 1. In addition to the figure caption, some comments are necessary. Straight bold lines represent averaged retarded and advanced GFs, given by (4) and its Hermitian conjugate. The squares built of GFs are Hikami boxes[19]. A

Hikami box (HB) is given by three contributions, cf.[15]: one empty box and two boxes with an impurity line inside. Wavy lines with (without) arrows represent cooperons (diffusons).

Using the identity $\sum_{\alpha=0}^{3} \tilde{\sigma}_{\alpha}^{s_1 s_3} \tilde{\sigma}_{\alpha}^{s_4 s_2} = 2\delta_{s_1 s_2}\delta_{s_3 s_4}$, we write the diffuson in the basis of matrices $\tilde{\sigma}$ as $D_{\mathbf{q}} = \frac{\hbar}{2m\tau}\left[\mathbb{1} - \tilde{X}_D(\mathbf{q})\right]^{-1}$, where $\mathbb{1}$ is a 4×4 unity matrix, and

$$\tilde{X}_D^{\alpha\beta}(\mathbf{q}) = \frac{\hbar}{2m\tau} \operatorname{Tr}_{\mathbf{p}}[\tilde{\sigma}_\alpha G_R^E(\mathbf{p})\tilde{\sigma}_\beta G_A^{E-\omega}(\mathbf{p} - \mathbf{q})]. \quad (10)$$

For cooperon lines in diagrams 1b and 1c, we use the expression $C_{\mathbf{q}} = \frac{\hbar}{2m\tau}\left[\mathbb{1} - \bar{X}_C(\mathbf{q})\right]^{-1}$, so that the cooperon series starts with *one* disorder line. Thus each diagram in Fig. 1 contains an extra contribution, belonging to the WL diagram. However, this contribution can affect S^r_{mn} in (9) only for $m \geq 1$, so we neglect it. To make the similarity between diffuson and cooperon more evident, we use a different matrix basis for the cooperon:

$$\bar{X}_C^{\alpha\beta}(\mathbf{q}) = \frac{\hbar}{2m\tau} \operatorname{Tr}_{\mathbf{p}}\{\bar{\sigma}_\alpha G_R^E(\mathbf{p})\bar{\sigma}_\beta^\dagger [G_A^{E-\omega}(\mathbf{q} - \mathbf{p})]^T\}, \quad (11)$$

where $\bar{\sigma}_\alpha = \sigma_2 \tilde{\sigma}_\alpha$, and $\sum_{\alpha=0}^{3} \bar{\sigma}_{\alpha}^{s_1 s_3} (\bar{\sigma}_{\alpha}^\dagger)^{s_4 s_2} = 2\delta_{s_1 s_2}\delta_{s_3 s_4}$. Because of the time-reversal symmetry, $\sigma_2 G_A^T(-\mathbf{p})\sigma_2 = G_A(\mathbf{p})$, so that $\bar{X}_C(\mathbf{q}) = \tilde{X}_D(\mathbf{q})$ and $C_{\mathbf{q}} = D_{\mathbf{q}}$. Thus the expressions for cooperon/diffuson lines are the same for all diagrams in Fig. 1, and the same is true for the lower HBs $L_{\alpha\beta\gamma}$. As a consequence, the sum of the three diagrams is equal to the diagram in Fig. 1a with a "renormalized" upper HB $R_{\alpha'\beta'\gamma'}$ (given by the sum of upper HBs of all three diagrams), see the expression for $\delta\sigma_{xx}^{\mathrm{II}}$ in (12).

Due to (C6) the anisotropy of the conductivity is given by $\sigma_{xx}(a, b) - \sigma_{yy}(a, b) = \sigma_{xx}(a, b) - \sigma_{xx}(a, -b)$, so it suffices to calculate only the SOI-dependent part of σ_{xx}. The total contribution of the diagrams in Fig. 1 is an integral from the product of two HBs (L and R) and three diffusons (D):

$$\delta\sigma_{xx}^{\mathrm{II}} = \frac{e^2}{h} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \sum_{\alpha,\beta,\gamma=0}^{3}$$
$$\sum_{\alpha',\beta',\gamma'=0}^{3} L_{\alpha\beta\gamma} D_{\mathbf{k}}^{\alpha\alpha'} D_{\mathbf{k+q}}^{\beta'\beta} D_{\mathbf{q}}^{\gamma\gamma'} R_{\alpha'\beta'\gamma'}. \quad (12)$$

First, we evaluate (12) for zero frequency $\omega = 0$. In the absence of SOI, the HBs are given by

$$L_{\alpha\beta\gamma}^0 = -2i\frac{p_F l\tau^3}{\hbar^4}(k_x + q_x) \operatorname{Tr}[\tilde{\sigma}_\alpha \tilde{\sigma}_\beta \tilde{\sigma}_\gamma],$$

$$R_{\alpha\beta\gamma}^0 = -4i\frac{p_F l\tau^3}{\hbar^4}\{(1 - \delta_{\beta 0})(k_x \delta_{\alpha 0}\delta_{\beta\gamma} + q_x \delta_{\gamma 0}\delta_{\alpha\beta}) +$$
$$(k_x + q_x)(1 - \delta_{\alpha 0})\left[\delta_{\beta 0}\delta_{\alpha\gamma} + i\epsilon_{\alpha\beta\gamma}(1 - \delta_{\beta 0})(1 - \delta_{\gamma 0})\right]\}, \quad (13)$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Cività tensor. The SOI-dependent part of the HBs is shown in Tab. I. Due to SOI, additional

α	0	0	1	1	2	2	3	3
β	1	3	1	3	1	3	1	3
γ	3	1	2	0	1	3	0	2
$\dfrac{L^{\mathrm{s}}_{\alpha\beta\gamma}\hbar^3}{2p_{\mathrm{F}}\tau^3(x_a+x_b)}$	-1	1	i	1	$-i$	$-i$	-1	i
$\dfrac{R^{\mathrm{s}}_{\alpha\beta\gamma}\hbar^3}{2p_{\mathrm{F}}\tau^3(x_a+x_b)}$	0	0	$-2i$	0	$2i$	$2i$	0	$-2i$

TABLE I: The first order SOI-correction to the Hikami boxes ($L^{\mathrm{s}}_{\alpha\beta\gamma}$ and $R^{\mathrm{s}}_{\alpha'\beta'\gamma'}$). Other matrix elements vanish. Each of three upper Hikami boxes has non-zero elements with $\alpha = 0$ or $\gamma = 0$, so that each diagram diverges at $\mathbf{k} \to 0$ or at $\mathbf{q} \to 0$; however, these elements cancel each other (note the zeroes in the last row), so that the complete expression for the conductivity converges.

nonzero matrix elements appear, $L^{\mathrm{s}}_{\alpha\beta\gamma}$ and $R^{\mathrm{s}}_{\alpha'\beta'\gamma'}$, so that $L_{\alpha\beta\gamma} = L^0_{\alpha\beta\gamma} + L^{\mathrm{s}}_{\alpha\beta\gamma}$ and $R_{\alpha'\beta'\gamma'} = R^0_{\alpha'\beta'\gamma'} + R^{\mathrm{s}}_{\alpha'\beta'\gamma'}$, see Tab. I.

Like the HBs, $\tilde{X}_D$ defined in (10) can be split into a main part and a SOI-dependent correction, which (for $\omega = 0$) are given by $\tilde{X}^0_D(\mathbf{q}) = \left(1 - l^2 q^2/2\hbar^2\right)\mathbb{1}$ and

$$\tilde{X}^{\mathrm{s}}_D(\mathbf{q}) = \tag{14}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(x_a+x_b)^2}{2} & 0 & -i\frac{q_x l}{\hbar}(x_a+x_b) \\ 0 & 0 & \frac{(x_a-x_b)^2}{2} & -i\frac{q_y l}{\hbar}(x_a-x_b) \\ 0 & i\frac{q_x l}{\hbar}(x_a+x_b) & i\frac{q_y l}{\hbar}(x_a-x_b) & x_a^2+x_b^2 \end{pmatrix}.$$

Hence the diffusons in (12) are given by

$$D_{\mathbf{q}} = \frac{\hbar}{2m\tau}\left[\mathbb{1} - \tilde{X}^0_D(\mathbf{q}) - \tilde{X}^{\mathrm{s}}_D(\mathbf{q})\right]^{-1}, \tag{15}$$

where $\tilde{X}^{\mathrm{s}}_D$ is written in (14).

Two angular integrations in (12) were performed analytically using `maxima`[20] (the result is too lengthy to be presented here), while the integrations over k, q were done numerically, using `quadpack`[21].

Finally, we obtain the following result for the anisotropy of the conductivity at $\omega = 0$,

$$\delta\sigma = 2\delta\sigma^{\mathrm{II}}_{xx} = \sigma_{xx} - \sigma_{yy} = 2S^2_{00}\frac{2x_a x_b}{x_a^2+x_b^2}\frac{e^2}{2\pi}\frac{1}{p_{\mathrm{F}}l}, \tag{16}$$
$$S^2_{00} = -5.6 \times 10^{-3}, \quad 2x_a x_b \ll x_a^2 + x_b^2 \ll 1.$$

The result (16) has been obtained in the CS rotated by $\pi/4$, where the sum of Rashba and Dresselhaus terms are given by V_s in (2). In the original coordinates, the conductivity has equal diagonal elements, and equal nonzero off-diagonal elements $\sigma_{xy} = \sigma_{yx} = \delta\sigma/2$.

The correction (16) depends non-analytically on the SOI in the vicinity of $x_a^2 + x_b^2 = 0$, so that at $\omega = 0$ even an infinitesimally small SOI can produce a finite anisotropy. This is not entirely surprising, since a similar non-analyticity is also seen to emerge in the weak-localization correction with Rashba SOI only and for vanishing dephasing (see, e.g., Eq. (17) in Ref.[7]).

The analyticity is restored at finite frequency:

$$\delta\sigma = -2 \cdot 0.25 \cdot \frac{-2i\omega\tau \cdot 2x_a x_b}{(x_a^2+x_b^2-2i\omega\tau)^2}\frac{e^2}{2\pi}\frac{1}{p_{\mathrm{F}}l}, \tag{17}$$
$$2x_a x_b \ll x_a^2 + x_b^2 \ll \omega\tau \ll 1.$$

To account for charge dephasing effects, we follow standard procedure[22] and substitute $-i\omega\tau \to \tau/\tau_\phi$, with τ_ϕ the charge dephasing time (e.g., due to electron-electron interactions). In addition, one can express the SOI parameters in terms of spin dephasing times of Dyakonov-Perel type[23], allowing us to rewrite (16), (17) as

$$\delta\sigma = \begin{cases} 5.6 \times 10^{-3} \cdot \dfrac{\tau_- - \tau_+}{\tau_- + \tau_+}\dfrac{e^2}{2\pi}\dfrac{1}{E_{\mathrm{F}}\tau}, & \tau_\pm \ll \tau_\phi, \\[2mm] 0.13 \cdot \left(\dfrac{\tau_\phi}{\tau_+} - \dfrac{\tau_\phi}{\tau_-}\right)\dfrac{e^2}{2\pi}\dfrac{1}{E_{\mathrm{F}}\tau}, & \tau_\phi \ll \tau_\pm, \end{cases} \tag{18}$$

where[23] $2\tau/\tau_\pm = (x_a \mp x_b)^2$. In a diffusive semiconductor (e.g., GaAs) for $p_{\mathrm{F}}l/\hbar = 5$ we estimate $\delta\sigma \lesssim 5 \cdot 10^{-4}\sigma_D$ in case $\tau_\pm \ll \tau_\phi$ and $\delta\sigma \lesssim 10^{-2}\sigma_D$ in the opposite case $\tau_\pm \gg \tau_\phi$.

Thus, in a system with finite dephasing time τ_ϕ, the anisotropic conductivity (18) is analytic in $\tau_\pm$. However, in a fully phase-coherent system with $\tau_\phi = \infty$ (which is commonly predicted at zero temperature), the conductivity tensor depends non-analytically on the SOI.

VI. CONCLUSION

The combination of Rashba and Dresselhaus SOI leads to anisotropy in the energy spectrum, which, in turn, leads to anisotropy in the conductivity of a disordered 2DEG. This anisotropy, being due to phase coherence of charge and spin, comes from two-loop diagrams in the perturbative approach, while it is absent in the Boltzmann equation approach. In the limit of full phase coherence, the system "jumps" from isotropic to anisotropic state for infinitesimally small SOIs.

APPENDIX A: CONDUCTIVITY FOR PURE-RASHBA OR PURE-DRESSELHAUS SOI

Using the time-reversal invariance we argued in the Introduction (Sec. I) that the conductivity tensor is isotropic in case of a pure-Rashba SOI. Here we demonstrate this isotropy for both pure-Rashba and pure-Dresselhaus cases using only rotation symmetry of the disorder-free part of the Hamiltonian.

The conductivity tensor in the rotated CS is connected with its value in the original CS through the transformation

$$\sigma_{\alpha\beta} \to (R_\phi \sigma R_\phi^{-1})_{\alpha\beta}, \tag{A1}$$

where R_ϕ is a 3D-rotation matrix in the (x, y)-plane by angle ϕ. In this section, we prove that when $a = 0$ or $b = 0$, the conductivity tensor is invariant with respect to rotations in the (x, y)-plane; together with the requirement (A1) this means that $\sigma_{\alpha\beta}$ is isotropic.

We define matrices

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (A2)$$

The Hamiltonian (2) can be rewritten as

$$\hat{H} = \frac{\hat{p}^2}{2m} + aV_R + bV_D + U(\mathbf{r}), \quad (A3)$$

where (in the original -unrotated- CS)

$$V_R = \mathbf{e}_z \cdot [\boldsymbol{\sigma} \times \hat{\mathbf{p}}], \quad V_D = \mathbf{e}_z \cdot [C\boldsymbol{\sigma} \times \hat{\mathbf{p}}], \quad (A4)$$

$\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$, and $\mathbf{e}_z$ is the unit vector along the z-axis. A rotation of the CS corresponds to the transformation of coordinates, momenta and spins according to

$$(\sigma_0, \boldsymbol{\sigma}) \to (\sigma_0, R_\phi \boldsymbol{\sigma}), \quad \hat{\mathbf{p}} \to R_\phi \hat{\mathbf{p}}, \quad \mathbf{r} \to R_\phi \mathbf{r}. \quad (A5)$$

We note that V_R is invariant under the transformation (A5):

$$V_R = \mathbf{e}_z \cdot [R_\phi \boldsymbol{\sigma} \times R_\phi \hat{\mathbf{p}}] = \begin{pmatrix} 0 & \hat{p}_y + i\hat{p}_x \\ \hat{p}_y - i\hat{p}_x & 0 \end{pmatrix}. \quad (A6)$$

The same invariance holds for the (disorder-averaged) pure-Rashba Green functions $G_{R/A}^{Ra}$. Thus, the contribution to the conductivity tensor from the (simplest) bubble diagram[15]

$$\sigma_{\alpha\beta}^{Ra} = \frac{e^2}{h} \operatorname*{Tr}_{\mathbf{p}} \operatorname*{Tr}_{\mathrm{spin}} \left[\hat{v}_\alpha^{Ra} G_R^{Ra} \hat{v}_\beta^{Ra} G_A^{Ra} \right],$$
$$\hat{v}_\alpha^{Ra} = \frac{i}{\hbar} \left[\frac{\hat{p}^2}{2m} + aV_R, r_\alpha \right] \quad (A7)$$

is invariant under the rotation (A5), and hence isotropic [due to A1)] in a pure-Rashba SOI system. As an example of a higher-loop correction, let us consider the two-loop contribution (12). The symmetry transformation (A5) will affect (12) twofold: (i) as a momentum rotation $\mathbf{k}, \mathbf{q} \to R_\phi \mathbf{k}, R_\phi \mathbf{q}$ and (ii) through a different set of Pauli matrices $(\sigma_0, \boldsymbol{\sigma}') = (\sigma_0, R_\phi \boldsymbol{\sigma})$ in the calculation of diffusons and Hikami boxes. The rotation of momenta can be absorbed by a variable change in the integration. Then, the result of the calculation does not depend on the choice of Pauli matrices, once they obey standard commutation relations. Since the new set $(\sigma_0, \boldsymbol{\sigma}')$ is connected through a unitary transformation with the standard one, $(\sigma_0, \boldsymbol{\sigma})$ the commutation relations are preserved so that (12) is invariant under the transformation (A5). Thus, the two-loop contribution (12) does not change when the CS is rotated in the (x, y)-plane. Analogously, the contribution

from any other diagram is invariant under the transformation (A5) in case when $b = 0$, so that [due to A1)] the pure-Rashba conductivity tensor is isotropic.

Let us now demonstrate that also in case of pure-Dresselhaus SOI the conductivity is isotropic. The Dresselhaus SOI can be transformed into Rashba SOI by a unitary transformation: $V_D = T^\dagger V_R T$; the same is true for the pure-Dresselhaus velocity operator: $\hat{v}_\alpha^{Db} = \frac{i}{\hbar} \left[\frac{\hat{p}^2}{2m} + bV_D, r_\alpha \right] = T^\dagger \hat{v}_\alpha^{Rb} T$. So, one can transform the pure-Dresselhaus conductivity tensor into the pure-Rashba one:

$$\sigma_{\alpha\beta}^{Db} = \frac{e^2}{h} \operatorname{Tr} \left[\hat{v}_\alpha^{Db} G_R^{Db} \hat{v}_\beta^{Db} G_A^{Db} \right]$$
$$= \frac{e^2}{h} \operatorname{Tr} \left[T^\dagger \hat{v}_\alpha^{Rb} T T^\dagger G_R^{Rb} T T^\dagger \hat{v}_\beta^{Rb} T T^\dagger G_A^{Rb} T \right], \quad (A8)$$

so that $\sigma_{\alpha\beta}^{Ra} = \sigma_{\alpha\beta}^{Da}$. With (A8) we proved this statement for the (simplest) bubble diagram[15]; it can be generalized to higher-loop corrections analogously to the pure-Rashba case [see (A7) above]. We conclude that the *disorder-averaged* conductivity tensor is isotropic in a system with either pure Rashba or pure Dresselhaus SOI.

APPENDIX B: CONDUCTIVITY IN CASE WHEN $a = \pm b$

The case, when the moduli of Rashba and Dresselhaus SOI amplitudes are equal, is special[15,24,25]. Both Hamiltonian and velocity operators can then be diagonalized in spin space by unitary transformations: for $a = b$, up to an irrelevant constant (in the rotated CS)

$$U_1 \hat{H} U_1^\dagger = \frac{m}{2} \left[\left(\frac{\hat{p}_x}{m} - 2a\sigma_3 \right)^2 + \frac{\hat{p}_y^2}{m^2} \right] + U(\mathbf{r}), \quad (B1)$$

and

$$U_1 \hat{\mathbf{v}} U_1^\dagger = \begin{pmatrix} \frac{\hat{p}_x}{m} - 2a\sigma_3 \\ \frac{\hat{p}_y}{m} \end{pmatrix}, \quad (B2)$$

where the transformation matrix is given by

$$U_1 = (U_1^\dagger)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\pi/4} \\ 1 & e^{i\pi/4} \end{pmatrix}. \quad (B3)$$

For $a = -b$,

$$U_2 \hat{H} U_2^\dagger = \frac{m}{2} \left[\left(\frac{\hat{p}_y}{m} + 2b\sigma_3 \right)^2 + \frac{\hat{p}_x^2}{m^2} \right] + U(\mathbf{r}), \quad (B4)$$

and

$$U_2 \hat{\mathbf{v}} U_2^\dagger = \begin{pmatrix} \frac{\hat{p}_x}{m} \\ \frac{\hat{p}_y}{m} + 2b\sigma_3 \end{pmatrix}, \quad (B5)$$

with

$$U_2 = (U_2^\dagger)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{-i\pi/4} \\ 1 & e^{-i\pi/4} \end{pmatrix}. \qquad \text{(B6)}$$

Let us denote the (unaveraged) Green functions without SOI ($a = b = 0$) as $\hat{g}_{r/a}$.

With $U_{1,2}$ from (B3) and (B6) we diagonalize all operators under the trace Tr in the conductivity expressions below. Then the spin part of $\text{Tr} \equiv \text{Tr}_{\text{spin}} \text{Tr}_{\mathbf{p}}$ splits into two terms. For $a = b$,

$$a = b \Rightarrow \underset{\mathbf{p}}{\text{Tr}} \underset{\text{spin}}{\text{Tr}} \left[\hat{v}_x \hat{G}_R \hat{v}_x \hat{G}_A \right]$$

$$= \underset{\mathbf{p}}{\text{Tr}} \underset{\text{spin}}{\text{Tr}} \left[U_1 \hat{v}_x U_1^\dagger U_1 \hat{G}_R U_1^\dagger U_1 \hat{v}_x U_1^\dagger U_1 \hat{G}_A U_1^\dagger \right]$$

$$= \underset{\mathbf{p}}{\text{Tr}} \left\{ \left[\frac{\hat{p}_x}{m} \hat{g}_r \frac{\hat{p}_x}{m} \hat{g}_a \right] \Big|_{\hat{p}_x \to \hat{p}_x - 2ma} + \left[\frac{\hat{p}_x}{m} \hat{g}_r \frac{\hat{p}_x}{m} \hat{g}_a \right] \Big|_{\hat{p}_x \to \hat{p}_x + 2ma} \right\}$$

$$= 2 \underset{\mathbf{p}}{\text{Tr}} \left[\frac{\hat{p}_x}{m} \hat{g}_r \frac{\hat{p}_x}{m} \hat{g}_a \right] = 2 \underset{\mathbf{p}}{\text{Tr}} \left[\frac{\hat{p}_y}{m} \hat{g}_r \frac{\hat{p}_y}{m} \hat{g}_a \right],$$
$$\text{(B7)}$$

and for $a = -b$,

$$a = -b \Rightarrow \underset{\mathbf{p}}{\text{Tr}} \underset{\text{spin}}{\text{Tr}} \left[\hat{v}_y \hat{G}_R \hat{v}_y \hat{G}_A \right]$$

$$= \underset{\mathbf{p}}{\text{Tr}} \underset{\text{spin}}{\text{Tr}} \left[U_2 \hat{v}_y U_2^\dagger U_2 \hat{G}_R U_2^\dagger U_2 \hat{v}_y U_2^\dagger U_2 \hat{G}_A U_2^\dagger \right]$$

$$= \underset{\mathbf{p}}{\text{Tr}} \left\{ \left[\frac{\hat{p}_y}{m} \hat{g}_r \frac{\hat{p}_y}{m} \hat{g}_a \right] \Big|_{\hat{p}_y \to \hat{p}_y - 2mb} + \left[\frac{\hat{p}_y}{m} \hat{g}_r \frac{\hat{p}_y}{m} \hat{g}_a \right] \Big|_{\hat{p}_y \to \hat{p}_y + 2mb} \right\}$$

$$= 2 \underset{\mathbf{p}}{\text{Tr}} \left[\frac{\hat{p}_y}{m} \hat{g}_r \frac{\hat{p}_y}{m} \hat{g}_a \right] = 2 \underset{\mathbf{p}}{\text{Tr}} \left[\frac{\hat{p}_x}{m} \hat{g}_r \frac{\hat{p}_x}{m} \hat{g}_a \right].$$
$$\text{(B8)}$$

Thus, we see that in the rotated CS $\sigma_{\alpha\alpha}$ are the same as in the absence of SOI for $a = \pm b$:

$$a = \pm b \implies \sigma_{xx} = \sigma_{yy} = \sigma_{xx}(a = b = 0). \quad \text{(B9)}$$

This proves that the ZLA result (7), as well as the Boltzmann equation result[12], are incomplete.

APPENDIX C: FURTHER SYMMETRIES OF $\sigma_{\alpha\beta}$

While the Rashba SOI V_R is invariant under rotations [see (A6)], the Dresselhaus SOI V_D does not possess this symmetry – rotation by an angle ϕ transforms it as follows:

$$V_D \to \mathbf{e}_z \cdot [C \boldsymbol{R}_\phi \sigma \times R_\phi \hat{\mathbf{p}}], \qquad \text{(C1)}$$

which is in general different from

$$V_D = \mathbf{e}_z \cdot [R_\phi C \boldsymbol{\sigma} \times R_\phi \hat{\mathbf{p}}]. \qquad \text{(C2)}$$

In the special case of rotation by $\phi = \pi/2$, the anticommutator

$$\{R_{\pi/2}, C\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \text{(C3)}$$

is zero in the (x, y)-plane, so that the r.h.s. of (C1) differs from (C2) only by the sign. On the other hand,

$$R_{\pi/2} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} R_{\pi/2}^{-1} = \begin{pmatrix} \sigma_{yy} & -\sigma_{yx} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}, \qquad \text{(C4)}$$

so that rotation by $\pi/2$ is equivalent to the sign change of the Dresselhaus SOI amplitude b, and

$$\sigma_{yy}(a, b) = \sigma_{xx}(a, -b), \quad \sigma_{xx}(a, b) = \sigma_{yy}(a, -b). \quad \text{(C5)}$$

Even without the disorder averaging one can see that only terms even in (a, b) may produce a non-zero result of Tr_{spin} in (1). Together with the property (C5) this leads to[26]

$$\sigma_{yy}(-a, b) = \sigma_{xx}(a, b) = \sigma_{xx}(-a, -b) = \sigma_{yy}(a, -b). \quad \text{(C6)}$$

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the "vertex correction" is given by a "bubble" where only one velocity operator is renormalized, see (7).

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